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Symmetry and Topology in Quantum Logic

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A *test space* is a collection of non-empty sets, usually construed as the catalogue of (discrete) outcome sets associated with a family of experiments. Subject to a simple combinatorial condition called *algebraicity*, a test space gives rise to a "quantum logic"—that is, an orthoalgebra. Conversely, all orthoalgebras arise naturally from algebraic test spaces. In non-relativistic quantum mechanics, the relevant test space is the set $\mathfrak{F}(\mathbf{H})$ of frames (unordered orthonormal bases) of a Hilbert space $\mathbf{H}$. The corresponding logic is the usual one, i.e., the projection lattice $L(\mathbf{H})$ of $\mathbf{H}$. The test space $\mathfrak{F}(\mathbf{H})$ has a strong symmetry property with respect to the unitary group of $\mathbf{H}$, namely, that any bijection between two frames lifts to a unitary operator. In this paper, we consider test spaces enjoying the same symmetry property relative to an action by a compact topological group. We show that such a test space, if algebraic, gives rise to a compact, atomistic topological orthoalgebra. We also present a construction that generates such a test space from purely group-theoretic data, and obtain a simple criterion for this test space to be algebraic.

KEY WORDS: quantum logics; symmetry; orthoalgebras; compact groups.

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1. INTRODUCTION

The primordial quantum logic is the orthomodular lattice $L(\mathbf{H})$ of projection operators on a separable Hilbert space $\mathbf{H}$. Familiar order-theoretic and partial-algebraic generalizations include orthomodular lattices, orthomodular posets, orthoalgebras, and effect algebras. But $L(\mathbf{H})$ is not *just* an order-theoretic object: it also has a rich topological and covariant structure. It would seem reasonable to study abstract quantum logics endowed with such structure. As a first observation, note that $L(\mathbf{H})$, in its norm topology, is *not* a topological lattice—it is easy to see that its meet and join operations are not continuous. However, $L(\mathbf{H})$ *is* a topological orthoalgebra in a natural sense.

This paper reviews and extends some recent work along these lines. After sketching a theory of (mainly, compact) topological orthoalgebras, following

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Wilce (2005a, 2005b), I will present a construction that produces highly symmetric compact topological orthoalgebras from group-theoretic data.

This depends on the representation of orthoalgebras as logics of test spaces. Recall that a *test space* (Foulis *et al.*, 1993) is a pair $(X, \mathfrak{A})$ where X is a non-empty set and $\mathfrak{A}$ is an irredundant covering of X by subsets, called *tests* (usually understood as outcome sets for various experiments). If $\mathfrak{A}$ satisfies a combinatorial constraint called *algebraicity*, one can construct an orthoalgebra $L(X, \mathfrak{A})$ that stands in roughly the same relationship to $(X, \mathfrak{A})$ that the projection lattice $L(\mathbf{H})$ of a Hilbert space $\mathbf{H}$ does to the set of orthonormal bases of $\mathbf{H}$. Indeed, if X is the unit sphere of $\mathbf{H}$ and $\mathfrak{A}$ is the set of orthonormal bases for $\mathbf{H}$, then $(X, \mathfrak{A})$ *is* algebraic, and $L(X, \mathfrak{A})$ is canonically isomorphic to $L(\mathbf{H})$. Now let E be a finite set—think of it as the outcome set for some "standard" measurement—and let S denote the group of all permutations of E. Let G be any group extending S, and let K be a subgroup of G with $K \cap S = S_{x_o}$ (the stabilizer of x_o in S). Then the action of G on the space $X := G/K$ of left K-cosets extends the action of S on E. Let $\mathfrak{A}$ denote the orbit of E in $\mathcal{P}(X)$. The test space $(X, \mathfrak{A})$ turns out to be algebraic precisely when, for every $A \subseteq E$, $F_A F_{E\setminus A} = F_{E\setminus A} F_A$, where F_A is the subgroup of G fixing each point of A. Moreover, if G is compact and K is a closed subgroup, the logic $L(X, \mathfrak{A})$ is in a natural way a compact, atomistic topological orthoalgebra, G acts naturally on L, and the space of atoms of L is a transitive G-space under this action.

2. BACKGROUND ON (TOPOLOGICAL) ORTHOALGEBRAS

An *orthoalgebra* (Foulis *et al.*, 1992) is a structure $(L, \oplus, 0, 1)$ consisting of a set L, two distinguished elements 0 and 1, and a commutative, associative, cancellative partial operation $\oplus$ such that, for all $a \in L$,

- $a \oplus 0 = a$;
- $\exists\, a'L$ with $a \oplus a' = 1$;
- $a \oplus a$ exists only if $a = 0$.

Any orthoalgebra L can be partially ordered by defining $a \leq b$ to mean that $b = a \oplus c$ for some $c \in L$. If it exists, c is unique; we denote it by $b \ominus a$. The mapping $a \mapsto a'$ is an orthocomplementation with respect to $\leq$, and $a \oplus b$ is defined iff $a \perp b$, i.e., $a \leq b'$. If defined, $a \oplus b$ is a *minimal* upper bound for $a, b, \in L$.

Proposition 2.1. (Foulis *et al.*, 1992) *Let* $(L, \oplus, 0, 1)$ *be an orthoalgebra. The following are equivalent:*

(a) $(L, \leq, ')$ *is an OMP;*
(b) $\forall\, a, b \in L, a \perp b \Rightarrow a \oplus b = a \vee b$;
(c) $\forall\, a, b, c \in L, a, b, c$ *pairwise orthogonal* $\Rightarrow (a \oplus b) \oplus c$ *exists.*

Condition (c) is called *ortho-coherence*. Thus, orthomodular posets can equivalently be described as ortho-coherent orthoalgebras, and orthomodular lattices, as lattice-ordered orthoalgebras.

If $F = \{x_1, \ldots, x_n\}$ is a finite subset of an orthoalgebra L, we say that F is *jointly orthogonal*, or *summable*, iff $\bigoplus F := x_1 \oplus x_2 \cdots \oplus x_n$ exists. We call an arbitrary subset A of L jointly orthogonal iff every finite set $F \subseteq A$ is jointly orthogonal. In this case, we define $\bigoplus A$ to be the join $\vee_{F \subseteq E, F_{\text{finite}}} \bigoplus F$, provided this exists. If every element of L has the form $\bigoplus A$ for some jointly orthogonal set A of *atoms* of L, we say that L is *atomistic*.

Definition 2.2. A *topological orthoalgebra* (TOA) is an orthoalgebra L equipped with a topology making the relation $\perp \subseteq L \times L$ closed, and the operations $\oplus : \perp \to L$ and $' : L \to L$, continuous.

One can show (Wilce, in press-b, Lemma 3.2) that if L is a TOA, the order relation $\leq$ is closed in $L \times L$, from which it follows that L is Hausdorff, and that the mapping $\ominus : \leq \to L$ is continuous. It is also worth noting that any compact TOA is order complete, in the sense that every upwardly-directed net has a supremum (Wilce, in press-b, Lemma 3.6). In particular, any compact, lattice-ordered TOA is a complete lattice.

Example 2.3.

(a) Any Cartesian product of discrete orthoalgebras, with the product topology, is a compact TOA.
(b) A *topological orthomodular lattice* (TOML), in the sense of Choe and Greechie (1993), is an orthomodular, Hausdorff topological lattice, in which the orthocomplementation $' : L \to L$ is continuous. Any TOML L yields a TOA, since in that setting $a \perp b$ iff $a \leq b'$ iff $a = a \wedge b'$—a closed relation, since L is Hausdorff and $\wedge, '$ are continuous.
(c) The projection lattice $L(\mathbf{H})$ of a Hilbert space $\mathbf{H}$ is a lattice-ordered TOA—but not a TOML—with respect to either the norm or strong (equivalently, weak) operator topology (Wilce, in press-b, Example 3.4).

As the example of $L(\mathbf{H})$ illustrates, a lattice-ordered TOA need not be a topological lattice. In view of this, the following result (Wilce, in press-b, Proposition 3.9) is interesting.

Proposition 2.4. *A compact Boolean TOA is a topological lattice, hence, a topological Boolean algebra.*

Proof: If L is any TOA, the set

$$\mathbf{M}(L) := \{(a, b, c) \in L^3 | c \leq a, c \leq b, \text{ and } a \ominus c \perp b\}$$

is closed in L^3. If L is Boolean, $\mathbf{M}(L)$ is the graph of the mapping $a, b \mapsto a \wedge b$. Thus, $\wedge$ has a closed graph. If L is compact, it follows that $\wedge$ is continuous. □

A subset of an OA L is said to be *compatible* if it is contained in a Boolean sub-OA of L. If every finite pairwise-compatible subset of L is compatible, L is *regular*.[2] Using Proposition 2.4, one can prove (Wilce, in press-b, Theorem 3.12) that in a compact, regular TOA, every block is a compact Boolean algebra. From this, it follows that such a TOA is atomistic.

Another condition that insures the atomicity of a compact TOA is that it have an isolated zero. Call a subset of a TOA L *totally non-orthogonal* iff it contains no two orthogonal elements. The following results are also from (Wilce, in press-b):

Lemma 2.5. *Every non-zero element of a TOA L has a totally non-orthogonal open neighborhood.*

Proof: If $a \in L$ is non-zero, then $(a, a) \notin \perp$. Since the relation $\perp$ is closed in L^2, we can find open sets U and V with $(a, a) \in U \times V$ and $(U \times V) \cap \perp = \emptyset$. The set $U \cap V$ is a totally non-orthogonal open neighborhood of a. □

Proposition 2.6. *Let L be a compact TOA with isolated zero. Then L is atomistic. Moreover, there exists a positive integer n such that every element of L is the orthogonal sum of at most n atoms.*

Proof: If L is compact with 0 isolated, then $L\backslash\{0\}$ is compact. By Lemma 2.5, we can cover it by finitely many totally non-orthogonal open sets $U_1, \ldots, U_n$. A pairwise-orthogonal subset of $L\backslash\{0\}$ meets U_i at most once, and so, has at most n elements. It follows that no element of L can be expressed as the orthogonal sum of more than n non-zero elements—whence, every element is the orthogonal sum of at most n atoms. □

3. CONSTRUCTING (TOPOLOGICAL) OAs

There is a standard method, due to D. J. Foulis and C. H. Randall, for constructing orthoalgebras from combinatorial structures called *test spaces*. We outline this below (further discussion and motivation can be found in the papers Foulis *et al.* (1992, 1993), or in the survey Wilce (2000)). We then show (following Wilce, in press-a) how the construction of orthoalgebras from test spaces can be topologized.

[2] Of course, regularity implies orthocoherence, so a regular orthoalgebra is the same thing as a regular OMP.

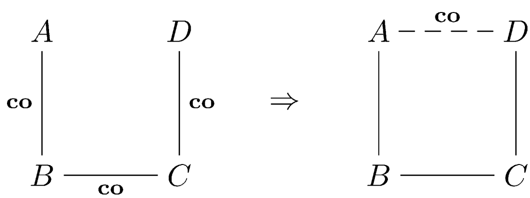
3.1. Background on Test Spaces

A *test space* $(X, \mathfrak{A})$ consists of a set X and a covering $\mathfrak{A}$ of X by non-empty subsets, called *tests*. These are usually understood as outcome sets for various experiments; accordingly, subsets of tests are called *events*. The *rank* of $(X, \mathfrak{A})$ is the least upper bound of the cardinalities of tests $E \in \mathfrak{A}$. If this is finite, we say that $(X, \mathfrak{A})$ has *finite rank*.

Let $\mathcal{E} = \mathcal{E}(X, \mathfrak{A})$ denote the set of all events of a test space $(X, \mathfrak{A})$. Call events $A, B \in \mathcal{E}$

- *orthogonal*, writing $A \perp B$, iff $A \cap B = \emptyset$ and $A \cup B \in \mathcal{E}$;
- *complementary*, writing A **co** B, iff $A \cap B = \emptyset$ and $A \cup B \in \mathfrak{A}$; and
- *perspective*, writing $A \sim B$, iff they have a common complementary event.

One calls a test space $(X, \mathfrak{A})$ *algebraic* iff perspective events have the same complementary events—equivalently, if every "hook" of events A **co** B **co** C **co** D *closes* with A **co** D, as illustrated diagrammatically below:



If $(X, \mathfrak{A})$ is algebraic, then the perspectivity relation $\sim$ is an equivalence relation on $\mathcal{E}$, and the quotient set $\Pi(X, \mathfrak{A}) := \mathcal{E}/\sim$ carries a partial operation

$$[A], [B] \mapsto [A] \oplus [B] := [A \cup B],$$

well defined for orthogonal events A and B. In fact, $(\Pi, \oplus)$ is an orthoalgebra, called the *logic* of $(X, \mathfrak{A})$. Any orthoalgebra L can be constructed in this way: set $X = L \backslash \{0\}$, and let $\mathfrak{A}$ be the collection of all finite *orthopartitions of unity* in L, that is, finite jointly orthogonal sets $E \subseteq X$ with $\bigoplus E = 1$. Then $(X, \mathfrak{A})$ is an algebraic test space, with $\Pi(X, \mathfrak{A})$ canonically isomorphic to L via the mapping $[A] \mapsto \bigoplus A$.

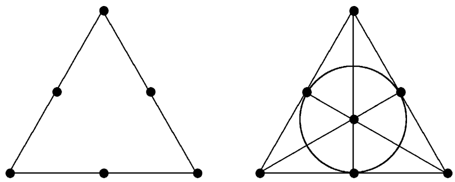
Let us now consider some examples.

3.1.1. Classical Test Spaces

The test spaces of interest in discrete classical probability theory consist of just a single test. Let $X = E$ and $\mathfrak{A} = \{E\}$. Then $\mathcal{E} = \mathcal{P}(E)$, and $A \sim B$ iff $A = B$. Hence, the logic $\Pi(X, \mathfrak{A})$ is just $\mathcal{P}(E)$, regarded as a Boolean orthoalgebra.

3.1.2. Two Non-Classical Test Spaces

Here are two simple, finite algebraic test spaces leading to non-Boolean orthoalgebras. (a) Let X consist of the nodes in the graph (i), below and let $\mathfrak{A}$ consist of the sets of nodes lying along straight lines. This is algebraic by default. The three corner nodes are pairwise, but not jointly, orthogonal; hence, Π is a non-orthocoherent orthoalgebra. (b) Let X consist of the points, and $\mathfrak{A}$ of the lines of the Fano plane, pictured below in graph (ii). Again $(X, \mathfrak{A})$ is algebraic, with a non-orthocoherent logic.



(i) (ii)

3.1.3. Quantum Test Spaces

In non-relativistic quantum theory, the relevant test spaces are as follows. Let $X = X(\mathbf{H})$ be the unit sphere of a Hilbert space $\mathbf{H}$—real or complex, finite or infinite dimensional—and let $\mathfrak{F} = \mathfrak{F}(\mathbf{H})$ be the collection of *frames* (unordered orthonormal bases) of $\mathbf{H}$. Then events of $(X, \mathfrak{F})$ are orthonormal subsets of $\mathbf{H}$; two events are perspective iff they have the same closed span, and complementary iff their closed spans are complementary subspaces of $\mathbf{H}$. Hence, $(X, \mathfrak{F})$ is algebraic, with $\Pi(X, \mathfrak{F}) \simeq L(\mathbf{H})$. We refer to $(X, \mathfrak{F})$ as the *frame test space* associated with $\mathbf{H}$. If phase relations are not important (in particular, if one is not considering multi-stage experiments), one can replace $X(\mathbf{H})$ with the set $P(\mathbf{H})$ of one-dimensional projection operators, and $\mathfrak{F}(\mathbf{H})$, with the collection $\mathfrak{B}(\mathbf{H})$ of maximal pairwise-orthogonal sets of such projections. The pair $(P, \mathfrak{B})$ is again an algebraic test space, which we will call the *projective test space* of $\mathbf{H}$, with logic isomorphic to the projection lattice of $\mathbf{H}$.

3.2. Topological Test Spaces

We can topologize the apparatus of test spaces and logics, as follows (Wilce, in press-a). Call two (distinct) outcomes $x, y \in X$ of a test space $(X, \mathfrak{A})$ *orthogonal*, and write $x \perp y$, iff $\{x, y\} \in \mathcal{E}$.

Definition 3.1. A *topological test space* is a test space $(X, \mathfrak{A})$ where

(a) X is a Hausdorff space, and
(b) the relation $\perp \subseteq X^2$ is closed.

It is straightforward that every outcome $x \in X$ has a totally non-orthogonal open neighborhood—the proof is essentially the same as that of Lemma 2.5. Using this, one can show that every event of a topological test space is a closed, discrete set (Wilce, in press-a, Proposition 2.2(d)). Arguing as in the proof of Proposition 2.6, one can also show that a compact topological test space has finite rank.

We regard the space $\mathcal{E} = \mathcal{E}(X, \mathfrak{A})$ of events of a topological test space $(X, \mathfrak{A})$, as a subspace of the hyperspace 2^X of all closed subsets of X, in the *Vietoris topology* (Illanes and Nadler, 1999). This is the weakest topology on 2^X making the sets

$$[U] = \{F \in 2^X | F \cap U \neq \emptyset\} \quad \text{and} \quad (U) = \{F \in 2^X | F \subseteq U\}$$

open for every open set $U \subseteq X$—or, equivalently, the weakest topology making $[U]$ open if U is open, and closed if U is closed.[3] Note that $\emptyset$ is isolated in 2^X.

In following sections of this paper, we shall be concerned mainly with compact, hence finite-rank, test spaces. The following notation and observations will prove useful. If $U_1, \ldots, U_n$ are open sets in X, let $\langle U_1, \ldots, U_n\rangle$ denote the Vietoris open set

$$[U_1] \cap \cdots \cap [U_n] \cap (U_1 \cup \cdots \cup U_n).$$

In other words, $\langle U_1, \ldots, U_n\rangle$ is the set of all closed sets consisting of at least one point from each of the sets $U_1, \ldots, U_n$. Notice that if $U_1, \ldots, U_n$ are pairwise disjoint, then any closed set belonging to $\langle U_1, \ldots, U_n\rangle$ must have at least n elements. It follows that, for every $k \in \mathbb{N}$, the set of closed sets $F \in 2^X$ with $|F| > k$ is open in 2^X.

Lemma 3.2. *If X is any Hausdorff space, let $F_k(X)$ denote the set of all non-empty finite subsets of* X *of size $\leq k$. Let $q : X^k \to F_k$(X) be the surjection given by $q : (x_1, \ldots, x_k) \mapsto \{x_1, \ldots, x_k\}$. Then, we have the following:*

(a) $F_k(X)$ is closed in 2^X.
(b) If $\mathcal{B}$ is a basis for the topology on X, then the sets $\langle U_1, \ldots, U_k\rangle \cap F_k$, where $U_1, \ldots, U_k$ are (not necessarily distinct) open sets in $\mathcal{B}$ with $\{U_1, \ldots, U_k\}$ pairwise disjoint, form a basis for F_k.
(c) q is an open continuous mapping, hence, a quotient mapping.

Proof: Parts (a) and (b) are well known and straightforward. Proofs can be found in Illanes and Nadler (1999). For part (c), let $U_1, \ldots, U_k$ be open subsets of X. Then

$$q(U_1 \times \cdots \times U_k) = \langle U_1, \ldots, U_k\rangle \cap F_k(X) \ ,$$

[3] If X is a compact metric space, this coincides with the topology on 2^X induced by the Hausdorff metric.

so q is an open mapping. Also, if $U_1, \ldots, U_k$ are pairwise disjoint, so that $\langle U_1, \ldots, U_k\rangle \cap F_k(X)$ is a basic open set in $F_k(X)$, then

$$q^{-1}(\langle U_1, \ldots, U_k\rangle \cap F_k(X)) = \bigcup_{\sigma} U_{\sigma(1)} \times \cdots \times U_{\sigma(k)}$$

where σ runs over all permutations of $\{1, 2, \ldots, k\}$; thus, q is continuous. □

Lemma 3.3. *In any topological test space, $\mathcal{E}_k$ is clopen in $\mathcal{E}$ in the latter's relative Vietoris topology.*

Proof: If $A = \{a_1, \ldots, a_k\}$ is any k-element event, let $V_1, \ldots, V_k$ be pairwise disjoint, totally non-orthogonal open sets with $a_i \in V_i$. Then $\mathcal{V} = \langle V_1, \ldots, V_k\rangle$ is a Vietoris open neighborhood of A in 2^X. Any *event* contained in $\mathcal{V}$ will be contained in $\bigcup_{i=1}^k V_i$, and will contain *exactly one* outcome from each V_i—hence, will have exactly k outcomes. Thus, $\mathcal{E}_k$ is open. To see that it is clopen, let $\mathcal{E}_{>k}$ denote the set of events having more than k outcomes. If $A \in \mathcal{E}_{>k}$, we can find pairwise disjoint open sets $U_1, \ldots, U_{k+1}$ with $A \in [U_1] \cap \cdots \cap [U_{k+1}] =: \mathcal{U}$. Any event—indeed, any closed set—$B \in \mathcal{U}$ must meet each set U_i. As these are pairwise disjoint, $|B| \geq k + 1$, whence, $B \in \mathcal{E}_{>k}$. Thus, $\mathcal{E}_{>k}$ is open. Thus, we have finitely many pairwise disjoint open sets $\{\emptyset\}, \mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_k, \mathcal{E}_{>k}$ partitioning $\mathcal{E}$. It follows that each of these sets is clopen in $\mathcal{E}$. □

3.3. Logics of Algebraic Topological Test Spaces

If a topological test space $(X, \mathfrak{A})$ is algebraic, we can give its logic $\Pi(X, \mathfrak{A})$ the quotient topology induced by the natural surjection $\mathcal{E} \to \Pi$, where $\mathcal{E} \subseteq 2^X$ has its relative Vietoris topology. We would like this to make it a topological orthoalgebra in the sense of Definition 2.2. The following result, essentially proved in Wilce (in press-a), gives some sufficient conditions for this to be so.

Call a topological test space $(X, \mathfrak{A})$ *stably complemented* if the set

$$\mathcal{U}^{\mathbf{co}} = \{B \in \mathcal{E}(X, \mathfrak{A}) | \exists A \in \mathcal{U}, A \,\mathbf{co}\, B\}$$

is open for every open set $\mathcal{U} \subseteq \mathcal{E}$.

Proposition 3.4. *Let $(X, \mathfrak{A})$ be a compact, stably complemented, algebraic topological test space with $\mathcal{E}$ closed in 2^X. Then $\Pi(X, \mathfrak{A})$ is a compact TOA with isolated zero.*

Proof: That $\Pi(X, \mathfrak{A})$ is a topological orthoalgebra is proved in Wilce (in press-a) (Proposition 3.6). There, it is also shown (Lemma 4.5) that if $(X, \mathfrak{A})$ is stably complemented, the canonical quotient mapping $\mathcal{E} \to \Pi$ is open. Since $\mathcal{E}$ is compact, so is 2^X in its Vietoris topology (Illanes and Nadler, 1999). Since $\mathcal{E}$ is closed in 2^X, it, too is compact. Thus, $\Pi = \mathcal{E}/\sim$ is compact. Since $\emptyset$ is isolated in 2^X, and so also in $\mathcal{E}$, it follows (since the quotient mapping $\mathcal{E} \to \Pi$ is open) that $0 = [\emptyset]$ is isolated in Π. □

The topological assumptions in Proposition 3.4 are quite strong. It would be interesting to know if they could be weakened. However, as we shall see in Section 5, they are automatically satisfied in the presence of sufficient symmetry.

4. SYMMETRIC TEST SPACES

The quantum-mechanical test space $(X_{\mathbf{H}}, \mathfrak{F}(\mathbf{H}))$ associated with a Hilbert space **H** is marked by a very high degree of symmetry. Indeed, if E and F are two orthonormal bases for **H**, then $|E| = |F|$, and any bijection $f : E \to F$ extends uniquely to a unitary operator on **H**. In this section, we consider topological test spaces having similar symmetry properties.

4.1. Background on G-Spaces

Let G be a topological group, and let K be a subgroup of G; let G/K be the space of left cosets of K, with the quotient topology, and let $\pi : G \to G/K$ be the canonical quotient mapping $\pi(\alpha) = \alpha K$. It is not difficult to show that π is an open mapping, and that G/K is Hausdorff iff K is closed in G. A *G-space* is a topological space X equipped with a continuous action $G \times X \to X$. If X is transitive—that is, for any $x, y \in X$, there exists some $\alpha \in G$ with $y = \alpha x$—then for any base point $x_o \in X$, we have a continuous surjection $f : G \to X$ given by $f : \alpha \mapsto \alpha x_o$. If $K \leq G$ is the stabilizer of x_o, then we have a canonical bijection $\phi : G/K \to X$ given by $\phi : \alpha K \mapsto \alpha x_o$. ϕ is continuous (since $\phi \circ \pi = f$, f is continuous, and π is open). Therefore, if G, and hence also G/K, is compact, and X is Hausdorff, ϕ is a homeomorphism, whence, $f : \alpha \mapsto \alpha x_o$ is open.

Definition 4.1. A *G-test space* is a topological test space $(X, \mathfrak{A})$ where X is a G-space and αE $\mathfrak{A}$ for every $E \in \mathfrak{A}$ and every $\alpha \in G$. A G-test space $(X, \mathfrak{A})$ is *symmetric* iff G acts transitively on $\mathfrak{A}$, and the stabilizer, G_E, of any test $E \in \mathfrak{A}$ acts transitively on E. We shall say that $(X, \mathfrak{A})$ is *fully symmetric* iff all tests have the same cardinality (finite or otherwise), and any bijection between two tests is effected by some element of G. If this element is always unique, then we shall say that $(X, \mathfrak{A})$ is *strongly* symmetric.

As noted earlier, the test space of frames of a Hilbert space **H** is strongly symmetric with respect to **H**'s unitary group $U(\mathbf{H})$. The projective space of **H** is fully, but not strongly symmetric with respect to $U(\mathbf{H})$ (since a bijection between two maximal orthogonal sets of one-dimensional projections determines a unitary operator only up to a choice of phase factors). The Fano plane test space of Example 3.1.2(b) is strongly symmetric with respect to its automorphism group (i.e., the collineation group of the Fano plane).

Notice that if $(X, \mathfrak{A})$ is G-symmetric, then X is a transitive G-space. If G is compact, it follows that X is homeomorphic to G/K, where K is the stabilizer of any point $x_o \in X$. In particular, X is compact; hence, $(X, \mathfrak{A})$ has finite rank.

Proposition 4.2. *Let $(X, \mathfrak{A})$ be a symmetric G-test space, where G is compact. Then the natural action of G on $\mathcal{E}(X, \mathfrak{A})$ is continuous.*

Proof: Let $\mathcal{E}^*$ denote the set of non-empty events of $(X, \mathfrak{A})$. Since the empty event is both invariant and isolated, it will be enough to show that G's action on $\mathcal{E}^*$ is continuous. Since G is compact, $(X, \mathfrak{A})$ has finite rank, say k. Thus, $\mathcal{E}^*$ is contained in the set $F_k(X)$ of finite subsets of X having k or fewer elements. By Lemma 3.2(c), the canonical surjection $q : X^k \to F_k(X)$ is continuous and open. Giving X^k the natural diagonal G-action, q is equivariant. It follows easily that the action of G on $F_k(X)$—and hence, on any invariant subset of $F_k(X)$, e.g., $\mathcal{E}^*$—is also continuous. □

All discrete symmetric test spaces can be recovered as instances of the following construction.

Construction 4.3. *Let E be a set (regarded, perhaps, as the outcome set for some "standard" experiment), and let H be a group acting transitively on E. Let G be any group extending H, and let $K \leq G$ be a subgroup extending the stabilizer H_{x_o} of some fixed element $x_o \in E$, with $H \cap K = H_{x_o}$. Let $X = G/K$. There is a natural H-equivariant injection $E \to X$ given by $\sigma x_o \mapsto \sigma K$, where $\sigma \in H$. Identifying E with its image under this injection, we may suppose that $E \subseteq X$. Now let $\mathfrak{A}$ denote the orbit of E under G's action on $\mathcal{P}(X)$: the pair $(X, \mathfrak{A})$ is then a G-symmetric test space. Conversely, given a G-symmetric test space $(X, \mathfrak{A})$, choose any test $E \in \mathfrak{A}$ and any outcome $x_o \in E$; setting $H = G_E$ and $K = G_{x_o}$ (the stabilizers, respectively, of E and x_o in G), the preceding construction reproduces $(X, \mathfrak{A})$.*

Remarks

(a) Note that we can begin Construction 4.3 with purely group theoretic data. Indeed, if G is a group and H, K are subgroups of G, set $X = G/K$ and let E be the orbit of K in X under the action of H, i.e., $E = \{\eta K | \eta \in H\} \subseteq X$. Setting $\mathfrak{A} = \{\alpha E | \alpha \in G\}$, as given earlier, $(X, \mathfrak{A})$ is a G-symmetric test space.

(b) If we take H to act as the full symmetric group S_E of all bijections on E, the resulting symmetric test space $(X, \mathfrak{A})$ will be fully symmetric. It will be *strongly* symmetric iff, in addition, the only element of G fixing every outcome in E is the identity element.

Thus far, our construction has yielded only a combinatorial object, that is, a discrete symmetric test space. We now show that if G is a topological group and H and K are closed subgroups with $H\backslash K$ closed, then the test space obtained from G, H and K, as given earlier, is a symmetric topological test space with respect to the quotient topology on $X = G/K$. We begin with a purely combinatorial

Lemma 4.4. *Let $(X, \mathfrak{A})$ be G-symmetric, let $x_o \in E \in \mathfrak{A}$ be given, and let $K = G_{x_o}$ and $H = G_E$, the stabilizers of x_o and E, respectively. For each $\alpha \in G$, let $x_\alpha = \alpha x_o$. Then, for all $\alpha, \beta \in G$, $x_\alpha \perp x_\beta$ iff $\beta^{-1}\alpha \in K(H\backslash K)K$.*

Proof: As $x_\alpha \perp x_\beta$ iff $x_{\beta^{-1}\alpha} \perp x_o$, it is sufficient to show that $x_\alpha \in x_o^\perp$ iff $\alpha \in K(H\backslash K)K$. Suppose first that $\alpha = \beta_\sigma \gamma$ where $\beta, \gamma \in K$ and $\sigma \in H\backslash K$. Then $x_o \perp \sigma x_o$ so $x_o = \beta x_o \perp \beta\sigma x_o = \beta\sigma\gamma x_o = \alpha x_o$. Conversely, suppose $x_\alpha \perp x_o$. Then $x_\alpha \neq x_o$, and there exists some test $E_\beta := \beta E \in \mathfrak{A}$ with $x_o, x_\alpha \in E_\beta$. It follows that there exist $\sigma, \sigma' \in H$ with (i) $x_\alpha = \beta\sigma x_o$ and (ii) $x_o = \beta\sigma' x_o$. From (ii), we have $\beta\sigma' \in K$, whence, $\beta \in K_{\sigma'_{-1}}$. Now (i) requires that $x_\alpha = \beta\sigma x_o \neq x_o$ so $\sigma^{'-1}\sigma \in H\backslash K$. We also have from (i) that $(\beta\sigma)^{-1}\alpha \in K$, whence, $\alpha \in \beta\sigma K \subseteq K\sigma^{'-1}\sigma K \subseteq K(H\backslash K)K$. □

Theorem 4.5. *Let G be a compact topological group and H, K two subgroups of G. Form the test space $(X, \mathfrak{A})$ as in Construction 4.3, with $X = G/K$ having the quotient topology. Then $(X, \mathfrak{A})$ is a topological test space iff both K and $H\backslash K$ are closed in G.*

Proof: As remarked earlier, G/K is Hausdorff iff K is closed in G. It remains to show that the orthogonality relation on X is closed in $X \times X$ iff $H\backslash K$ is closed in G. Notice that, since G is compact and acts continuously on both X and $\mathfrak{A}$, both of the stabilizers $K = G_{x_o}$ and $H = G_{E_o}$ are compact, hence, closed. If $H\backslash K$ is closed, then certainly so is $K(H\backslash K)K$ (as this is the image of the compact set $K \times (H\backslash K) \times K$ under the continuous mapping $(\alpha, \beta, \gamma) \mapsto \alpha\beta\gamma$). Thus, so is the set $\{(\alpha, \beta) | \beta^{-1}\alpha \in K(H\backslash K)\}$. Finally, since G is compact, the image of this set under the quotient mapping $(\alpha, \beta) \mapsto (x_\alpha, y_\beta)$ is closed. But this image is just the orthogonality relation on X. For the converse, suppose $\perp$ is closed. Then so is $K(H\backslash K)K$, by Lemma 4.4. It follows that $(H\backslash K)$ is likewise closed. For suppose $\eta_i \to \eta$ in H, with $\eta_i \notin K$. If $\eta \in H \cap K$, then we have $\eta^{-1}\eta_i\eta \to \eta$ and $\eta^{-1}\eta_i\eta \in K(H\backslash K)K$, whence, $\eta \in K(H\backslash K)K$. Thus, we can find $\phi, \psi \in K$ and $\eta' \in H\backslash K$ with $\eta = \phi\eta'\psi$. Then $\eta' = \phi^{-1}\eta\psi^{-1} \in K$, a contradiction. □

Notice that the condition that $H\backslash K$ be closed will certainly hold if H is discrete. This is the case, for instance, for the frame test space of an n-dimensions

Hilbert space **H** with respect to $U(n)$, since here the stabilizer of an orthonormal basis E is isomorphic to the group of permutations of E.

5. FULLY SYMMETRIC TEST SPACES

If $(X, \mathfrak{A})$ is fully G-symmetric, then G acts transitively on each of the sets $\mathcal{E}_k$ of k-element events. To see this, suppose $A, B \in \mathcal{E}_k$: choose tests $E \supseteq A$ and $F \supseteq B$ and a bijection $f : A \to B$. Since $|E| = |F|$, we can extend f to a bijection $\bar{f} : E \to F$; by assumption, this is induced by a group element $\alpha \in G$. But then $\alpha A = B$. It follows from this, together with the general remarks on G-spaces at the beginning of Section 4, that, for each $A \in \mathcal{E}$, the mapping $G \mapsto \mathcal{E}_k$ given by $\alpha \mapsto \alpha A$, is continuous and open.

Theorem 5.1. *Let $(X, \mathfrak{A})$ be fully G-symmetric, with G compact. Then $(X, \mathfrak{A})$ is stably complemented, and $\mathcal{E}$ is closed in 2^X.*

Proof: As G is compact, $(X, \mathfrak{A})$ has finite rank, say rank n. By Lemma 3.3, each set $\mathcal{E}_k$ of k-element events, $k = 0, \ldots, n$, is clopen in $\mathcal{E}$, it suffices to show that, for every $k = 0, \ldots, n$, if $\mathcal{U}$ is open in $\mathcal{E}_k$, then $\mathcal{U}^{\mathbf{co}}$ is open in $\mathcal{E}_{n-k}$. As noted earlier, the mapping $G \to \mathcal{E}_k$ given by $\alpha \mapsto \alpha A$ is continuous and open for each $A \in \mathcal{E}_k$. Thus, if $\mathcal{U}$ is an open neighborhood of an event $A \in \mathcal{E}_k$, then the set $U = \{\alpha \in G | \alpha A \in \mathcal{U}\}$ is open in G. Let B **co** A. Then for every $\alpha \in U, \alpha B$ **co** $\alpha A \in \mathcal{U}$, i.e., $\alpha B \in \mathcal{U}^{\mathbf{co}}$. In other words, the open set $U \cdot B = \{\alpha B | \alpha \in U\}$ about B is contained in $\mathcal{U}^{\mathbf{co}}$. Thus, $\mathcal{U}^{\mathbf{co}}$ is open in $\mathcal{E}_{n-k}$.

It remains to show that $\mathcal{E}$ is closed in 2^X. It will suffice to show that each clopen set $\mathcal{E}_k$ is closed in $F_k(X)$ (since, by Lemma 3.2(a), the latter is closed in 2^X). Suppose, then, that A_i is a net in $\mathcal{E}_k$ converging in $F_k(X)$ to a set A. Since G acts transitively on $\mathcal{E}_k$, we can find a net α_i in G with $A_i = \alpha_i A_o$, where A_o is some arbitrary "base" event in $\mathcal{E}_k$. Since G is compact, we can choose a convergent subnet $\alpha_{i'} \to \alpha \in G$. By the continuity of the map $G \to F_k(X)$ given by $\alpha \mapsto \alpha A_o$, we have $A_{i'} = \alpha_{i'} A_o \to \alpha A_o \in \mathcal{E}$, in the latter's Vietoris topology. Since 2^X is Hausdorff, it follows that $A = \alpha A_o \in \mathcal{E}$. □

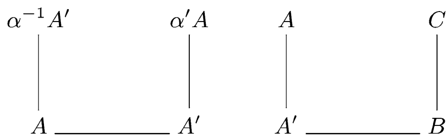
Thus, the topological assumptions of Proposition 3.4 are automatically satisfied for any fully G-symmetric test space with G compact. If $(X, \mathfrak{A})$ is algebraic, its follows that its logic $L = \Pi(X, \mathfrak{A})$ is a compact TOA with isolated zero—hence, in particular, that L is atomistic. Indeed, the atoms of L are precisely the points of the form $p(\{x\})$, where $x \in X$. It is easy to see that G continues to act on L by continuous automorphisms, and that the atoms of L form a transitive G-space.

The question remains: When *is* a fully G-symmetric test space algebraic?

Theorem 5.2. *Let $(X, \mathfrak{A})$ be a fully-symmetric G-test space. Choose and fix $E \in \mathfrak{A}$. If $A \subseteq E$, write A' for $E \backslash A$, and let F_A be the subgroup of G fixing each $x \in A$. Then $(X, \mathfrak{A})$ is algebraic iff, for every $A \subseteq E$, $F_A F_{A'} = F_{A'} F_A$.*

Proof: (⇒) Suppose $(X, \mathfrak{A})$ is algebraic. If $\alpha \in F_A$ and $\alpha' \in F_{A'}$, we obtain a "hook" of events $\alpha^{-1}A'$ **co** A **co** A' **co** $\alpha' A$ (see graph (i) later). Since $(X, \mathfrak{A})$ is algebraic, $\alpha^{-1}A'$ **co** $\alpha' A$. Let $\alpha^{-1}A' \cup \alpha' A =: F \in \mathfrak{A}$. Since $|\alpha^{-1}A'| = |A'|$ and $|\alpha' A| = |A|$, and since every bijection $E \to F$ extends to an element of G, we can find $\beta \in G$ with $\beta x = \alpha^{-1}x$ for every $x \in A'$ and $\beta x = \alpha' x$ for every $x \in A$. Then $\alpha\beta \in F_{A'}$ and $\alpha'^{-1}\beta F_A$—whence, $\beta^{-1}\alpha' \in F_A$ as well. Thus, $\alpha\alpha' = (\alpha\beta)(\beta^{-1}\alpha') \in F_{A'}F_A$. Thus, $F_A F_{A'} \subseteq F_{A'} F_A$.

(⇐) Now suppose that $F_A F_{A'} = F_{A'} F_A$ for every $A \subseteq E$. To show $(X, \mathfrak{A})$ is algebraic, it is sufficient to consider configurations of the form A **co** A' **co** B **co** C, with $A \subseteq E$, as in graph (ii) later (as any hook in $\mathcal{E}$ is a translate of one of these). We wish to show that A **co** C. Now, $B = \alpha' A$ for some $\alpha' \in F_{A'}$, and $C = \beta A'$ for some $\beta \in F_B$. But $F_B = F_{\alpha' A} = \alpha' F_A \alpha'^{-1} \subseteq F_{A'} F_A F_{A'}$. Since $F_{A'} F_A = F_A F_{A'}$, we have $F_{A'} F_A F_{A'} \subseteq F_A F_{A'}$. Thus, $\beta \in F_B \Rightarrow \beta = \alpha\alpha''$ where $\alpha \in F_A$ and $\alpha'' \in F_{A'}$. But then $C = \beta A' = \alpha A'$—whence, A **co** C. □



(i) (ii)

Example 5.3. As an illustration of the preceding result, let $G = U(\mathbf{H})$, the unitary group of a Hilbert space $\mathbf{H}$, and let E be an orthonormal basis for $\mathbf{H}$. If $A \subseteq E$, let [A] be the subspace spanned by A. Then F_A is the group of unitaries of the form $W = 1_{[A]} \oplus U$, where $1_{[A]}$ is the identity operator on $[A]$ and U is any unitary operator on $[A]^{\perp}$. Likewise, $F_{A'}$ consists of unitaries of the form $W' = V \oplus 1_{[A]^{\perp}}$, V a unitary on [A]. Since $WW' = W'W$ for any two such W and W', we have $F_A F_{A'} = F_{A'} F_A$.

5.1. Problems for Further Study

Here, as in the earlier papers Wilce (in press-a, in press-b), I have tried to make a case for the study of what may be called topological quantum structures. A great deal remains to be done. For instance, it would be good to know how much of the theory sketched here can be made to work without compactness assumptions. In particular, referring to Proposition 2.4: *need a non-compact Boolean TOA be a topological Boolean algebra*?[4]

In a different direction, Theorem 5.2 suggests the project of classifying, for a given compact group G, all fully G-symmetric algebraic topological test spaces of

[4] Added in proof: The answer is no. John Harding has recently constructed an ingenious counter example.

a given finite rank. It would be especially interesting to have such a classification for compact Lie groups.

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